Algebra Preliminary Examination
September 2016

Answer all questions.

1. a) Is a quotient of a UFD by a prime ideal necessarily a UFD? Give a proof or a counterexample.

b) Is a quotient of a PID by a prime ideal necessarily a PID? Give a proof or a counterexample.

2. Suppose R is a Noetherian integral domain. Let K be its fraction field, considered as an R-module in the natural way. Prove that an R-submodule $M \subset K$ is finitely-generated if and only if there is some nonzero $a \in R$ such that $aM \subset R$.

3. Prove that any localization of a commutative Noetherian ring is Noetherian.

4. Suppose K is a field containing a primitive nth root of 1, where $(n, \operatorname{char} K) = 1$. Prove that for any $a \in K$, the extension $K(\sqrt[n]{a})$ is Galois and cyclic over K.

5. Find the Jacobson radical of $M_2(K \otimes_k K)$ where $k = \mathbf{F}_2(x)$ and $K = \mathbf{F}_2(x^{\frac{1}{2}})$.

6. Find a representation of a finite group over a field and a subrepresentation that is not a direct summand.